UP ER BOUND METHOD FOR SINTERED POWDER MATERIALS IN PLANE STRAIN①

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ABSTRACT

Yield condition of sintered powder materials are introduced briefly, the velocity discontinuity law of which in plane strain is deduced and equation of upper bound power for plane strain is derived. The upper bound solution of pressure for sintered powder materials in smooth plate upsetting is derived from the upper bound method, and theoretical calculation is verified by experiment of smooth plate upsetting with rectangular specimen of sintered copper.

Key words: sintered powder material upper bound method plane strain

1 INTRODUCTION

With the development of plastic working technology of sintered powder materials and the wide employment of high performance parts of the materials, more attention is paid to the research of plastic mechanics of sintered powder materials. In order to obtain load, stress and velocity distribution of plastic working, slip line method for sintered powder materials is developed and applied to resolve plane strain problem[1]. But for this method construction of slip line field is difficult. Upper bound method for sintered powder materials is a less difficult method, in Ref. [1] the equation of upper bound power for $\sigma_m \leqslant 0$ is derived from the character of slip line field, and upper bound solution of extrusion pressure is given. But in Ref. [1] the equation of upper bound power for $\sigma_m > 0$ is not derived and the upper bound solution is not verified by experiment. These and other related problems of upper bound method for sintered powder materials in plane strain are researched.

2 BASIC EQUATION

2.1 *Yield Condition*

In principal stress coordinates (σ_1, σ_2, σ_3), yield condition[1~3] of sintered powder materials is:

$$\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] + (\alpha\sigma_m)^2 = Y^2 \qquad (1)$$

where α — yield coefficient of hydrostatic pressure, $\alpha = 3\sqrt{(1-\rho^2)/(2+\rho^2)}$; Y — equivalent yield strength of sintered powder material.

The geometric surface of equation (1) is an elliptic sphere. In order to simplify calculation, a hexagonal pyramid[1] in the elliptic sphere is used to express the yield surface of sintered powder material approximately. Then the yield condition[1,3] corresponding to the hexagonal pyramid is:

$$\left.\begin{array}{l} \sigma_m \leqslant 0, \\ \sigma_{max} - \sigma_{min} - (\sigma_{max} + \sigma_{min})\sin\theta = Y\cos\theta \\ \sigma_m > 0, \\ \sigma_{max} - \sigma_{min} + (\sigma_{max} + \sigma_{min})\sin\theta = Y\cos\theta \end{array}\right\} \qquad (2)$$

where σ_{max}, σ_{min} — the maximum and the minimum principal stresses in algebraic value respectively; θ — function of coefficient α, $\theta = \mathrm{tg}^{-1}(\alpha/2)$.

Apparently, yield surfaces of equation (1) and (2) are all convex. If relative density $\rho = 1$, i. e. sintered powder material changed into fully dense metal, then $\alpha = \theta = 0$, equations (1) and (2) are changed into Mises and Tresca yield condition respectively.

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2.2 *Yield Stress and Strain*

In plane strain state, medium principal stress is normal to deformation plane, the other two principal stresses of the deformation plane are the maximum and the minimum principal stress respectively. If take $\sigma_1 = \sigma_{max}$, $\sigma_2 = \sigma_{min}$, then equation (2) can be expressed by yield function as follows:

$$f = \sigma_1 - \sigma_2 \mp (\sigma_1 + \sigma_2)\sin\theta - Y\cos\theta = 0 \quad (3)$$

where minus and plus signs of "$\mp$" are corresponding to $\sigma_m \leqslant 0$ and $\sigma_m > 0$ respectively. This regulation is observed in the following paragraph.

From plastic potential theory[1], we can obtain principal strain rates $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ as follows:

$$\left.\begin{aligned} \dot{\varepsilon}_1 &= \dot{\lambda}\frac{\partial f}{\partial \sigma_1} = \dot{\lambda}(1 \mp \sin\theta) \\ \dot{\varepsilon}_2 &= \dot{\lambda}\frac{\partial f}{\partial \sigma_2} = \dot{\lambda}(-1 \mp \sin\theta) \end{aligned}\right\} \quad (4)$$

where $\dot{\lambda}$ is a non-negative constant.

The strain rates of equation (4) can be shown in Mohr's strain rate circles as Fig. 1, from which we have strain rate components $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, $\dot{\gamma}_{xy}$ as:

$$\left.\begin{aligned} \dot{\varepsilon}_x &= \dot{\lambda}(\cos 2\varphi \mp \sin\theta) \\ \dot{\varepsilon}_y &= \dot{\lambda}(-\cos 2\varphi \mp \sin\theta) \\ \dot{\gamma}_{xy} &= \dot{\lambda}\sin 2\varphi \end{aligned}\right\} \quad (5)$$

where φ is angle rotated from coordinate x to the maximum strain rate $\dot{\varepsilon}_1$ or the maximum stress σ_1, take the angle counter-clockwise as positive.

In coordinates (σ, τ), geometric curve of yield condition (2) can be shown as Fig. 2. The stress state expressed by Mohr's stress circle tangent to yield curve in Fig. 2 agrees with yield condition (3), and its stress components σ_x, σ_y, τ_{xy} are:

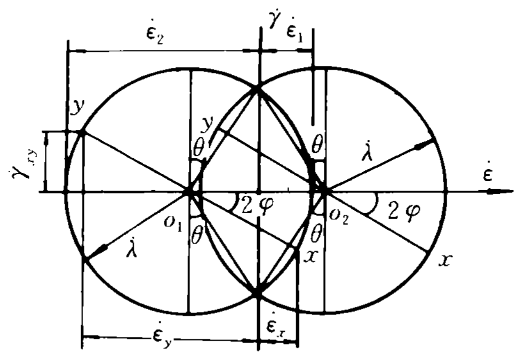

Fig. 1 Mohr's strain rate circle

$$\left.\begin{aligned} \sigma_x &= \sigma + R\cos 2\varphi \\ \sigma_y &= \sigma - R\cos 2\varphi \\ \tau_{xy} &= R\sin 2\varphi \end{aligned}\right\} \quad (6)$$

where σ —average stress, $\sigma = (\sigma_1 + \sigma_2)/2$; R —radius of Mohr's stress circle, from geometric relation of Fig. 2 we have:

$$R = Y/2\cos\theta \pm \sigma\sin\theta \quad (7)$$

3 VELOCITY DISCONTINUITY AND UPPER BOUND POWER EQUATION

3.1 *Velocity Discontinuity*

Velocity discontinuity is that velocity interruption takes place across discontinuous line. In fact discontinuous velocity line is a narrow zone with

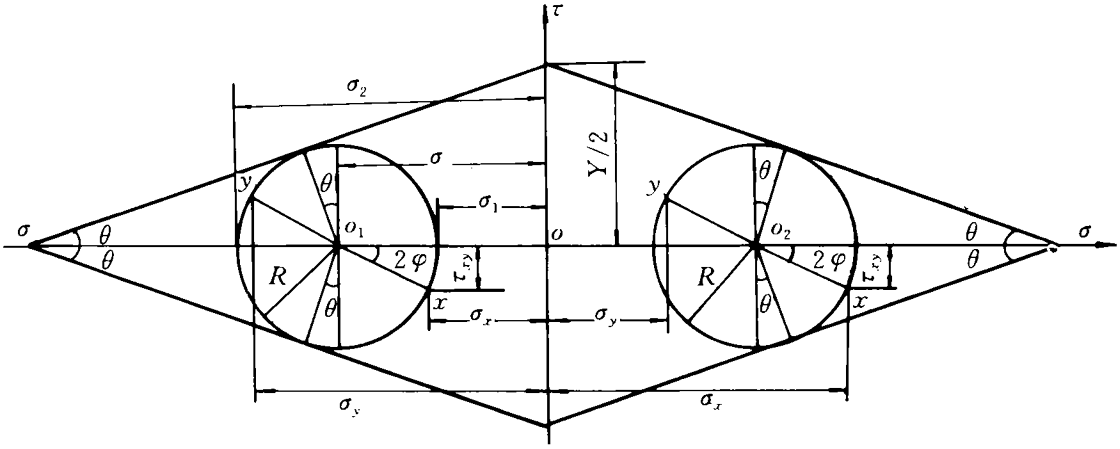

Fig. 2 Yield condition and Mohr's stress circle

thickness b shown in Fig. 3, if $b \to 0$, this zone is changed into a discontinuous velocity line. Take v_{t1}, v_{t2} and v_{n1}, v_{n2} as tangential and normal velocities on each side of discontinuous velocity line respectively. From expression (5) and geometric equation of deformation zone we have strain rate components $\dot{\varepsilon}_t$, $\dot{\varepsilon}_n$, $\dot{\gamma}_{tn}$ on discontinuous velocity line as follows:

$$\left.\begin{aligned}
\dot{\varepsilon}_t &= \dot{\lambda}(\cos 2\varphi_1 \mp \sin\theta) = \frac{\partial v_t}{\partial t} \\
\dot{\varepsilon}_n &= \dot{\lambda}(-\cos 2\varphi_1 \mp \sin\theta) \\
&= \frac{\partial v_n}{\partial n} = \frac{v_{n2} - v_{n1}}{b} \\
\dot{\gamma}_{tn} &= \dot{\lambda}\sin 2\varphi_1 = \frac{1}{2}\left(\frac{\partial v_t}{\partial n} + \frac{\partial v_n}{\partial t}\right) \\
&= \frac{1}{2}\left(\frac{v_{t2} - v_{t1}}{b} + \frac{\partial v_n}{\partial t}\right) \\
&\approx \frac{1}{2}\,\frac{v_{t2} - v_{t1}}{b}
\end{aligned}\right\} \qquad (8)$$

where φ_1 is angle rotated from discontinuous line to the maximum strain rate or the maximum stress.

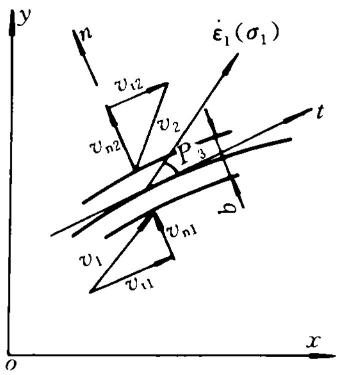

Fig. 3 Discontinuous velocity and discontinuous velocity line

If $b \to 0$, then from the second equation of expression (8) $\dot{\varepsilon}_n \to \infty$, $\dot{\lambda} \to \infty$. However $\frac{\partial v_t}{\partial t}$ i. e. $\dot{\varepsilon}_t$ is taken place in one side of discontinuous velocity line (not across the line), its value is finite, hence from the first equation of expression (8) we have $\cos 2\varphi_1 \mp \sin\theta = 0$, i. e.:

$$|\varphi_1| = \pi/4 \mp \theta/2 \qquad (9)$$

Substitute expression (9) into (8), we have $\dot{\varepsilon}_t = 0$, $\dot{\varepsilon}_n = \mp 2\dot{\lambda}\sin\theta = \frac{v_{n2} - v_{n1}}{b}$, $|\dot{\gamma}_{tn}| = \dot{\lambda}\cos\theta = \frac{1}{2}\left|\frac{v_{t2} - v_{t1}}{b}\right|$; hence:

$$\frac{\Delta v_n}{\Delta v_t} = \frac{v_{n2} - v_{n1}}{v_{t2} - v_{t1}} = \frac{\dot{\varepsilon}_n}{2\dot{\gamma}_{tn}} = \frac{\mp 2\dot{\lambda}\sin\theta}{2\dot{\lambda}\cos\theta} = \mp \operatorname{tg}\theta \qquad (10)$$

Expression (10) indicates discontinuous velocity inclines angle θ to discontinuous velocity line for sintered powder materials in plane strain. Take Δv as discontinuous velocity, then:

$$\left.\begin{aligned}
|\Delta v_t| &= |\Delta v|\cos\theta \\
|\Delta v_n| &= |\Delta v|\sin\theta
\end{aligned}\right\} \qquad (11)$$

If relative density $\rho = 1$, i. e. sintered powder material changed into fully dense metal, then $\theta = 0$, from expression (11) we have $\Delta v_n = 0$, which is the same as that of fully dense metal.

3.2 *Equation of Upper Bound Power*

Take V as volume of deformation body, S as total surface, and S is divided into velocity boundary S_v and stress boundary S_F. On S_v and S_F and velocity v_i and stress T_i are given respectively. The practical internal stress and strain rate field of deformation body are σ_{ij} and $\dot{\varepsilon}_{ij}$ respectively. Assume a kinematically admissible velocity field v_i^*, which is accorded with mass constancy and velocity boundary condition, and the internal stress field, strain rate field and discontinuous velocity surface of deformation body corresponding to the kinematically admissible velocity field v_i^* are σ_{ij}^*, $\dot{\varepsilon}_{ij}^*$ and S_D^* respectively. On S_D^* discontinuous velocity is Δv_i^*. If neglect volume force, then the equation [5] of upper bound power in plastic deformation is:

$$\int_{S_F} T_i v_i \, ds \leqslant \int_V \sigma_{ij}^* \dot{\varepsilon}_{ij}^* \, dV + \int_{S_D} T_i \Delta v_i^* \, ds - \int_{S_v} T_i v_i^* \, ds \qquad (12)$$

In plane strain, strain rate field $\dot{\varepsilon}_{ij}^*$ determined by stress field σ_{ij}^* corresponding to kinematically admissible velocity field v_i^* and accorded with yield condition (2) is meeted the case of expression (4), hence

$$\sigma_{ij}^* \dot{\varepsilon}_{ij}^* = \sigma_1^* \dot{\varepsilon}_1^* + \sigma_2^* \dot{\varepsilon}_2^* = \sigma_1^* \dot{\lambda}(1 \mp \sin\theta) + \sigma_2^* \dot{\lambda}(-1 \mp \sin\theta) = \dot{\lambda} Y \cos\theta$$

From expression (4) $\dot{\lambda} = \frac{\dot{\varepsilon}_1^* - \dot{\varepsilon}_2^*}{2} = \dot{\gamma}_{max}^*$, then

$$\int_V \sigma_{ij}^* \dot{\varepsilon}_{ij}^* \, dV = \int_V \dot{\gamma}_{max}^* Y \cos\theta \, dV \qquad (13)$$

See Fig. 3, take τ_{tn} and σ_n as shear and normal stresses, then

$$T_i\triangle v_i^* = |\tau_{tn}\triangle v_t^*| + |\sigma_n\triangle v_n^*|$$

Substitute expression (6) and (11) into above and consider expression (9), we have:

$$T_i\triangle v_i^* = |R\triangle v^*\cos^2\theta| + |(\sigma \mp R\sin\theta)\triangle v^*\sin\theta|$$

For $R > 0$, then $\sigma < 0$, $\sigma - R\sin\theta < 0$; $\sigma > 0$, then $\sigma + R\sin\theta > 0$; hence:

$$T_i\triangle v_i^* = R\triangle v^*\cos^2\theta \mp (\sigma \mp R\sin\theta)\triangle v^*\sin\theta = \frac{Y}{2}\triangle v^*\cos\theta \text{, therefore:}$$

$$\int_{S_D^*} T_i\triangle v_i^* ds = \int_{S_D^*} \frac{Y}{2}\triangle v^*\cos\theta ds \qquad (14)$$

Substitute expressions (13) and (14) into (12), we have equation of upper bound power for sintered powder materials in plane strain as follows:

$$\int_{S_u} T_i v_i ds \leqslant \int_V \dot{\gamma}^*_{max} Y\cos\theta dV + \int_{S_D^*} \frac{Y}{2}\triangle v^*\cos\theta ds - \int_{S_F} T_i v_i^* ds \qquad (15)$$

From expression (15) we see the equation of upper bound power corresponding to $\sigma_m \leqslant 0$ and $\sigma_m > 0$ is the same. If take deformation zone as a group of rigid blocks which slide each other, then $\dot{\gamma}^*_{max} = 0$; Furthermore if there is no stress on S_F then $\int_{S_F} T_i v_i^* ds = 0$, therefore equation (15) of upper bound power can be simplified as:

$$\int_{S_u} T_i v_i ds \leqslant \int_{S_D^*} \frac{Y}{2}\triangle v^*\cos\theta ds \qquad (16)$$

If relative density $\rho = 1$ i. e. the sintered powder material changed into fully dense metal, then $\theta = 0$, expressions (15), (16) are changed into equations[5] of upper bound power of fully dense metal in plane strain.

4 UPPER BOUND SOLUTION OF SMOOTH PLATE UPSETTING

Smooth plate upsetting of rectangular preform made from sintered powder material is shown in Fig. 4(a), the height of specimen is $2h$, the width of rigid smooth plate is $2W$, and two plates move towards the central plane with unit speed. If we divide deformation zone into a series of rigid blocks, then the boundary lines between the blocks are discontinuous velocity lines, which incline an angle β to the press plate. There is velocity discontinuity both in tangential and normal directions along the discontinuous line i. e. discontinuous velocity inclines angle θ to the discontinuous line. Because of symmetric deformation, it is enough to calculate upper bound power of quarter deformation zone in upper right of the preform. The hodagraph corresponding to the quarter zone is shown in Fig. 4(b). Take N as intersected points of central line and discontinuous lines, from Fig. 4(a) we can obtain length L of each discontinuous line as follows:

$$L = W/(N\cos\beta) \qquad (17)$$

From Fig. 4(b) we have discontinuous velocity $\triangle v^*$ as follows:

$$\triangle v^* = \frac{1}{\sin(\beta + \theta)} \qquad (18)$$

If take p as average pressure of plate, then from equation (16) of upper bound power we have

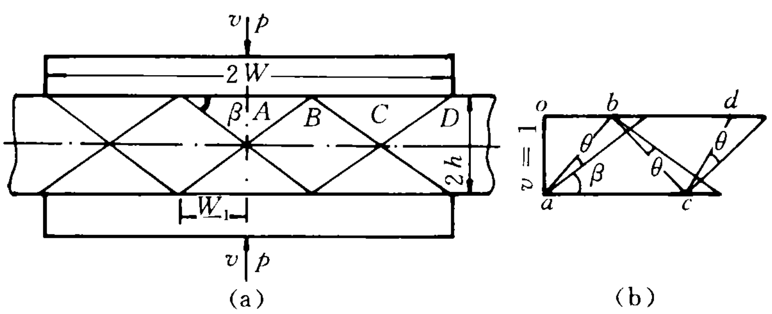

Fig. 4 Smooth plate upsetting of sintered powder material and the hodagraph

$$p \cdot W \cdot 1 \leqslant X(Y/2)\cos\theta\Delta v^{*} L$$
$$= \frac{Y}{2}\frac{W\cos\theta}{\cos\beta\sin(\beta+\theta)}$$
$$p \leqslant \frac{Y}{2}\frac{\cos\theta}{\cos\beta\sin(\beta+\theta)} = p(\beta) \qquad (19)$$

Differentiate expression (19) we have:

$$\frac{\partial p}{\partial \beta} = -\frac{Y}{2}\frac{\cos\theta\cos(2\beta+\theta)}{\cos^2\beta\sin^2(\beta+\theta)}$$

Make $\frac{\partial p}{\partial \beta} = 0$ we have $\beta = (\frac{\pi}{4} - \frac{\theta}{2})$; substitute it into expression (19) we have the minimum plate pressure as follows:

$$p = p(\frac{\pi}{4} - \frac{\theta}{2}) = \frac{Y\cos\theta}{1+\sin\theta} \qquad (20)$$

See Fig. 4(a), $W_1 = \frac{W}{X}$, $\frac{h}{W_1} = \mathrm{tg}\beta$, $X = \frac{W}{W_1} = \frac{W}{h}\mathrm{tg}\beta$. When $\beta = \frac{\pi}{4} - \frac{\theta}{2}$, the discontinuous velocity line is in accord with slip line[1,3], and the minimum upper bound solution is the same as slip line solution, furthermore in this case $\frac{W}{h}\mathrm{tg}(\frac{\pi}{4} - \frac{\theta}{2})$ is a positive integer. But If $\frac{W}{h}\mathrm{tg}(\frac{\pi}{4} - \frac{\theta}{2})$ is not an integer i. e. $\beta \neq (\frac{\pi}{4} - \frac{\theta}{2})$, then the upper bound solution can not be obtained from differentinating expression (19). In this case we rearrange expression (19) as follows:

$$p \leqslant \frac{Y}{2}\frac{\cos\theta}{\cos\beta\sin(\beta+\theta)}$$
$$= \frac{Y}{\sin^2\beta + 2\cos^2\beta\mathrm{tg}\theta}$$
$$= \frac{Y}{\frac{2}{\sin\beta + \mathrm{ctg}\beta} + \frac{2\mathrm{tg}\theta}{1+\mathrm{tg}^2\beta}} \qquad (21)$$

If take $\frac{W}{h} = r$, then from Fig. 4(a) we have $\mathrm{tg}\beta = \frac{X}{r}$, $\mathrm{ctg}\beta = \frac{r}{X}$, substitute them into expression (21), we have:

$$p \leqslant \frac{Y}{\frac{2}{\frac{X}{r} + \frac{r}{X}} + \frac{2\mathrm{tg}\theta}{1+(\frac{X}{r})^2}} \qquad (22)$$

Expression (22) is an upper bound solution corresponding to $\frac{W}{h}\mathrm{tg}(\frac{\pi}{4} - \frac{\theta}{2})$ not as an integer. In order to obtain lower pressure from expression (22), a proper value of X which makes $\beta = \mathrm{tg}^{-1}\frac{X}{r} \approx (\frac{\pi}{4} - \frac{\theta}{2})$ is needed to be determined. If take pressure of expression (20) as unit, then the upper bound plate pressure p which is a function of width-height ratio W/h and θ corresponding to relative density ρ is shown as Fig. 5. If relative density $\rho = 1$ i. e. sintered powder material changed into fully dense metal, then $\theta = 0$, expression (22) and the curve ($\theta = 0$) in Fig. 5 are changed into upper bound solutions[6] of fully dense metal.

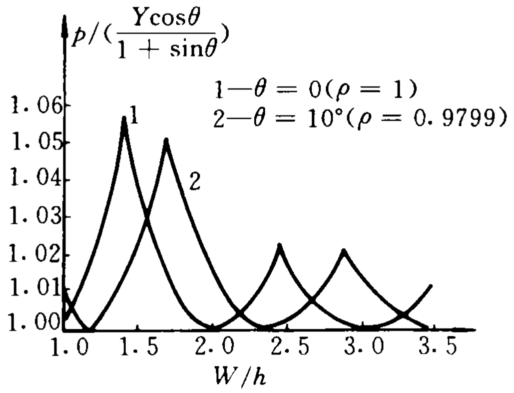

Fig. 5 Curve of upper bound plate pressure to width-height ratio and relative density

5 EXPERIMENTAL RESULTS

The experiment of rigid smooth plate upsetting of rectangular preform made from sintered copper is made to verify the upper bound calculation. Rectangular specimens are made from electrolytic powder copper (purity of copper ≥ 99.8%) by a process of blending, compacting and sintering. Sintering temperature and time are 920 ± 10 ℃ and 2 h respectively, sintering atmosphere is cracked ammonium. The experiment is conducted on a WI-60 type material test machine at room temperature. In order to minimize friction on the contact surface between specimen and tool and simulate frictionless smooth plate upsetting, the contact surfaces of specimen and tool are ground and polished and smeared with a lubricating zinc stearate-alcohol paste in test. Each specimen is compressed many times, and we record the load and its geometric dimension in each compression. The experimental results and theoretical calculations are shown in Table 1, the initial data of specimens are shown in Table 2. In

Table 1 Theoretical calculations and experimental results

Compressing number	No. 1					No. 2					No. 3				
	ρ	$\frac{W}{h}$	p_e/MPa	p_t/MPa	$\frac{p_t - p_e}{p_e}$%	ρ	$\frac{W}{h}$	p_e/MPa	p_t/MPa	$\frac{p_t - p_e}{p_e}$%	ρ	$\frac{W}{h}$	p_e/MPa	p_t/MPa	$\frac{p_t - p_e}{p_e}$%
1	0.823	1.65	94.4	109.0	15.5	0.805	1.51	57.3	56.3	−1.7	0.831	1.51	52.6	61.3	16.5
2	0.848	1.84	144.0	156.4	8.6	0.811	1.80	131.2	115.5	8.1	0.861	1.78	150.1	155.1	3.3
3	0.871	2.08	188.9	191.0	2.7	0.860	1.98	168.5	176.9	5.0	0.881	1.97	189.6	195.5	3.1
4	0.896	2.36	228.3	227.2	−0.5	0.895	2.11	232.5	225.5	−3.0	0.901	2.20	221.7	231.1	2.8
5	0.914	2.67	261.6	248.1	−5.2	0.911	2.68	259.1	211.2	5.9	0.922	2.45	259.9	258.8	0.4
6	0.925	3.02	287.9	263.8	−8.4	0.916	3.12	297.1	262.9	−11.6	0.947	3.02	315.1	299.0	−5.2

Notation: ρ, p_e, p_t are instant relative density, experimental and theoretical plate pressure respectively.

calculation of plate pressure the following equivalent yield strength Y of sintered copper is needed,

Table 2 Initial data of specimens

Specimen No	Weight /g	Length /mm	Width /mm	Height /mm	Initial relative density
1	143.2	44.90	15.09	29.65	0.801
2	142.8	44.87	15.08	29.63	0.800
3	143.1	44.31	14.94	29.26	0.830

which is determined[3] from uniaxial compression of cylindrical specimen as follows:

$$Y = A(\sqrt{1 + \frac{\alpha^2}{4}} + \frac{\alpha}{2})$$

$$\times (\ln \frac{\rho}{\rho_0} \sqrt{\frac{1 - \rho_0^2}{1 - \rho^2}})^n$$

$$A = -196.81 + 656.69\rho_0 \quad \text{(MPa)}$$

$$n = 0.9301 - 0.6171\rho_0$$

where ρ_0 is initial relative density of sintered copper.

From Table 1, experimental and theoretical plate pressures increase with increasing relative density and width-height ratio. Because friction on contact surfaces of specimen and die can not be eliminated thoroughly, and the friction increases with increasement of relative density and width-height ratio in upsetting, the experimental plate pressure increases faster than that of theoretical calculation corresponding to frictionless smooth plate upsetting, the relative error($p_t - p_e)/p_e$ changes from positive to negative in the upsetting. The absolute value of relative error is 10% more or less. This calculating precision is admissible for engineering design.

6 CONCLUSIONS

(1) Discontinuous velocity inclines an angle θ to discontinuous velocity line for sintered powder materials in plane strain is deduced.

(2) Equation of upper bound power for sintered powder materials in plane strain is derived.

(3) Upper bound pressure for smooth plate upsetting of sintered powder materials is obtained, and compared with experimental result of smooth plate upsetting of sintered copper.

REFERENCES

1 大矢根守哉，田端强. 塑性と加工，1974，15(156)：43－51.

2 Doraivelu, S M *et al*. Int J Mech Sci, 1984, 26(9/10): 527－535.

3 Zhao, Zhongzhi *et al*. Int J Sci of Sintering, 1987, 19(2): 65－80.

4 Wang, Ren *et al*. The foundations of plastic mechanics. Beijing: Science Press, 1982. 153－154.

5 Johnson, W; Mellor, P B. Plasticity Theory. Shanghai: Shanghai Science & Technology Press, 1965. 285－289.

6 Lippmann, H. Engineering Plasticity Theory for Metal Forming. Beijing: Metallrugical Industry Press, 1988. 238－240.